MATHEMATICS FUNDAMENTALS

EXPONENTS:

$$a^n = \prod_{x=1}^{n} a$$

$\{n \in \mathbb{N} \mid n > 0\}$

$a^{m+n} = a^m a^n$

$(ab)^n = a^n b^n$

$a^{m/n} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}$

$a^{m-n} = \dfrac{a^m}{a^n}$

$a^{-n} = \dfrac{1}{a^n}$

$(a^m)^n = a^{mn}$

LOGARITHMS:

$\log_b x = y$
$b^y = x$

$\log_b b = 1$
$\log_b 1 = 0$

$b^{\log_b x} = x$
$\log_b b^x = x$

Product Identity: $\log_b xy = \log_b x + \log_b y$
Quotient Identity: $\log_b \frac{x}{y} = \log_b x - \log_b y$
Power Identity: $\log_{b^q} x^p = \frac{p}{q} \log_b x$
Root Identity: $\log_b \sqrt[p]{x} = \frac{\log_b x}{p}$

Change of Base Formula: $\log_b x = \frac{\log_k x}{\log_k b}$
Natural Logarithm: $\log_e x = \ln x$
Common Logarithm: $\log_{10} x = \log x$

ABSOLUTE VALUE ($\forall a$):

$$|a| = \begin{cases} -a, & x < 0 \\ a, & x \geq 0 \end{cases}$$

$|-a| = |a|$
$|ab| = |a||b|$

$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$
$|a + b| \leq |a| + |b|$

BASIC FACTORING FORMULAS:

$x^2 - a^2 = (x - a)(x + a)$
$x^2 + a^2 = (x - ia)(x + ia)$
$x^2 + 2ax + a^2 = (x + a)^2$
$x^2 - 2ax + a^2 = (x - a)^2$
$x^2 + (a + b)x + ab = (x + a)(x + b)$
$x^3 + 3ax^2 + 3a^2x + a^3 = (x + a)^3$

$x^3 - 3ax^2 + 3a^2x - a^3 = (x - a)^3$
$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$
$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$
$x^n - a^n = (x - a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + ax^{n-2} + x^{n-1})$

S.K.C.

COMPLEX NUMBERS ($\mathbb{C}$):

Imaginary Unit: $i = \sqrt{-1}$ $i^2 = -1$ $\sqrt{-a} = i\sqrt{a}$

Euler Form ($\mathbf{a}, \mathbf{b} \in \mathbb{R}, \mathbf{b} \neq \mathbf{0}$): $\delta = a + bi \in \mathbb{C}$ $\boldsymbol{a}$**:** real part $\boldsymbol{b}$**:** imaginary part

Complex Modulus: $|\delta| = \sqrt{a^2 + b^2}$

Complex Argument\Phase: $\theta = \tan^{-1}\frac{b}{a}$

Phasor\Component Form ($\mathbf{z}, \boldsymbol{\theta} \in \mathbb{R}$): $\delta = |\delta|(\cos\theta + i\sin\theta) = |\delta| \operatorname{cis}\theta = |\delta|e^{\theta i}$ $\boldsymbol{\theta}$**:** complex argument\phase

Complex Conjugate of $\boldsymbol{\delta}$: $\overline{\delta} = a - bi$

Complex Addition: $(a + bi) + (c + di) = (a + c) + i(b + d)$

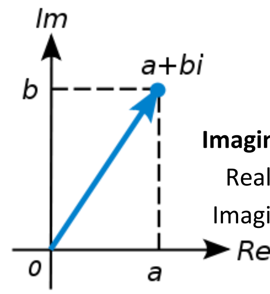
Complex Subtraction: $(a + bi) - (c + di) = (a - c) + i(b - d)$

Complex Multiplication: $(a + bi)(c + di) = (ac - bd) + i(ad + bc)$

Complex Division: $\frac{a+bi}{c+di} = \frac{(ac-bd)+i(bc-ad)}{c^2+d^2}$

De' Moivre's Identity: $\delta^n = |\delta|(\cos n\theta + i\sin n\theta)$

Complex Exponentiation: $a + bi^{c+di} = (a^2 + b^2)^{\frac{c+i}{2}} e^{i\tan^{-1}\frac{b}{a}(c+id)}$



Imaginary Plane
Real Axis vs.
Imaginary Axis

LIMITS:

($f(x) \rightarrow Real\ function; a, b, c, S, L \in \mathbb{R}; I \rightarrow Open\ interval\ containing\ c$)

$$\lim_{x \to c} f(x) = L \leftrightarrow \forall x \in I\backslash\{c\}, f(x) \in \mathbb{R} \land \forall \varepsilon > 0\ \exists\ \delta > 0 : \forall x\ (0 < |x - c| < \delta \rightarrow |f(x) - L| < \epsilon)$$

Limit as $\boldsymbol{x}$ Approaches $+\infty$: $\lim_{x \to +\infty} f(x) = L \leftrightarrow \forall \varepsilon > 0\ \exists\ S > 0 : \forall x\ (x > S \rightarrow |f(x) - L| < \varepsilon)$

Limit as $\boldsymbol{x}$ Approaches $-\infty$: $\lim_{x \to -\infty} f(x) = L \leftrightarrow \forall \varepsilon > 0\ \exists\ S < 0 : \forall x\ (x < S \rightarrow |f(x) - L| < \varepsilon)$

Limit Approaching $+\infty$: $\lim_{x \to c} f(x) = +\infty \leftrightarrow \forall x \in I\backslash\{c\}, f(x) \in \mathbb{R} \land \forall \varepsilon > 0\ \exists\ \delta > 0 : \forall x\ (0 < |x - c| < \delta \rightarrow f(x) > \varepsilon)$

Limit Approaching $-\infty$: $\lim_{x \to c} f(x) = -\infty \leftrightarrow \forall x \in I\backslash\{c\}, f(x) \in \mathbb{R} \land \forall \varepsilon < 0\ \exists\ \delta > 0 : \forall x\ (0 < |x - c| < \delta \rightarrow f(x) < \varepsilon)$

Limit from the Right: $\lim_{x \to c^+} f(x) = L \leftrightarrow \forall x \in I\backslash\{c\}, f(x) \in \mathbb{R} \land \forall \varepsilon > 0\ \exists\ \delta > 0 : \forall x\ (0 < x - c < \delta \rightarrow |f(x) - L| > \varepsilon)$

Limit from the Left: $\lim_{x \to c^-} f(x) = L \leftrightarrow \forall x \in I\backslash\{c\}, f(x) \in \mathbb{R} \land \forall \varepsilon > 0\ \exists\ \delta > 0 : \forall x\ (0 < c - x < \delta \rightarrow |f(x) - L| < \varepsilon)$

Existence of a Limit: $\lim_{x \to c} f(x) = L \leftrightarrow \lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L$

Given that all limits are defined and $k, c \in \mathbb{R}$:

Constant Law: $\lim_{x \to c} k = k$,

Identity Law: $\lim_{x \to c} x = a$

Scalar Law: $\lim_{x \to c}[kf(x)] = k \lim_{x \to c} f(x)$

Sum and Difference Law: $\lim_{x \to c}[f(x) \pm g(x)] = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$

Product Law: $\lim_{x \to c}[f(x)g(x)] = \lim_{x \to c} f(x) \lim_{x \to c} g(x)$

Quotient Law: $\lim_{x \to c}[f(x)/g(x)] = \lim_{x \to c} f(x) / \lim_{x \to c} g(x)$, given $\lim_{x \to c} g(x) \neq 0$

Power Law: $\lim_{x \to c} f^n(x) = \left(\lim_{x \to c} f(x)\right)^n$, given $\{n \in \mathbb{N} \mid n > 0\}$

Root Law: $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)}$

L'Hospital's Rule: If $\lim_{x \to c} \frac{f(x)}{g(x)} = \begin{cases} \frac{\pm\infty}{\pm\infty} \\ \frac{0}{0} \end{cases} \rightarrow \lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$, given $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists and $g'(x) \neq 0$

Squeeze Theorem: $\forall x \in I \backslash \{c\}, f(x), g(x), h(x) \in \mathbb{R} \wedge g(x) \leq f(x) \leq h(x) \wedge \lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L \rightarrow \lim_{x \to c} f(x) = L$

$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$ $\lim_{n \to \infty} \left(1 + \frac{\mathrm{a}}{n}\right)^n = e^a$ $\lim_{n \to \infty} \left(\frac{e^n - 1}{n}\right)^n = 1$

CONTINUITY AND DISCONTINUITY:

At a point c: $f(x)$ is continuous at $c \leftrightarrow f(c) \in \mathbb{R} \wedge \lim_{x \to c} f(x) \in \mathbb{R} \wedge f(c) = \lim_{x \to c} f(x)$

On an open interval $(a.b)$: $f(x)$ is continuous on $(a, b) \leftrightarrow f(x)$ is continuous $\forall x \in (a, b)$

On a closed interval $[a, b]$: $f(x)$ is continuous on $(a, b) \leftrightarrow f(x)$ is continuous $\forall x \in (a, b)$, $\lim_{x \to a^+} f(x) = f(a)$ (i.e. right-continuous), and $\lim_{x \to b^-} f(x) = f(b)$ (i.e. left-continuous)

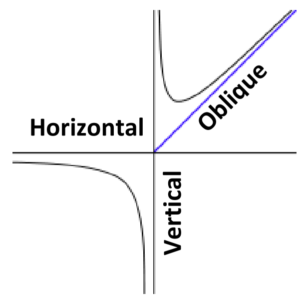
Removable Discontinuity: a removable discontinuity exists at $x = c \leftrightarrow \lim_{x \to c} f(x) \in \mathbb{R} \wedge f(x)$ is not continuous at c

Essential Discontinuity: an essential discontinuity exists at $x = c \leftrightarrow \lim_{x \to c} f(x) \notin \mathbb{R} \wedge f(x)$ is not continuous at c

Vertical Asymptote: $y = f(x)$ has the vertical asymptote $x = c \leftrightarrow$
$\lim_{x \to c^-} f(x) = \pm\infty \vee \lim_{x \to c^+} f(x) = \pm\infty$ regardless of $f(c)$

Horizontal Asymptote: $y = f(x)$ has the horizontal asymptote $y = c \leftrightarrow$
$\lim_{x \to +\infty} f(x) = c \vee \lim_{x \to -\infty} f(x) = c$

Oblique Asymptote: $y = f(x)$ has the oblique asymptote $y = mx + b$ $(m \neq 0)$ $c \leftrightarrow$
$\lim_{x \to +\infty} f(x) - (mx + b) = 0 \vee \lim_{x \to -\infty} f(x) - (mx + b) = 0$



Intermediate Value Theorem: $f(x)$ is continuous on $[a, b]$ and u is between $f(a)$ and $f(b) \rightarrow \exists c \in [a, b] : f(c) = u$

DERIVATIVES:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} = \frac{df(x)}{dx} = D_x f(x)$$

The derivative of y with respect to x ($y = f(x)$) provided all limits exist.

$$f'(c) = y' \mid_{x=c} = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \left.\frac{df(x)}{dx}\right|_{x=c} = D_x f(c)$$

The derivative of y with respect to x ($y = f(x)$) evaluated at c provided all limits exist.

Differentiability: Given $y = f(x)$, $\lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(a)}{\Delta x}$ exists $\forall c$ in the domain of $f \rightarrow f(x)$ is a differentiable function.

Differentiability at a Point: Given $y = f(x)$, $\lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(a)}{\Delta x}$ exists $\rightarrow f(x)$ is differentiable at $x = c$.

Differentiability at point $x = a$ implies continuity at point $x = a$

Given that $f(x)$ and $g(x)$ are differentiable functions and $a, b, c \in \mathbb{R}$:

Second Order Derivative of $y = f(x)$: $y'' = f''(x) = \left(f'(x)\right)' = \frac{d^2y}{dx^2} = \frac{d^2f}{dx^2}(x) = \frac{d^2}{dx^2}f(x) = D_{x^2}y$

Third Order Derivative of $y = f(x)$: $y''' = f'''(x) = \left(\left(f'(x)\right)'\right)' = \frac{d^3y}{dx^3} = \frac{d^3f}{dx^3}(x) = \frac{d^3}{dx^3}f(x) = D_{x^3}y$

nth Order Derivative of $y = f(x)$: $y^{(n)} = f^{(n)}(x) = \frac{d^ny}{dx^n} = \frac{d^nf}{dx^n}(x) = \frac{d^n}{dx^n}f(x) = D_{x^n}y$

Constant Rule: $c' = 0$

Constant Factor Rule (Kutz Rule): $\left(cf(x)\right)' = cf'(x)$

Sum Rule: $\left(f(x) + g(x)\right)' = f'(x) + g'(x)$

Subtraction Rule: $\left(f(x) - g(x)\right)' = f'(x) - g'(x)$

Linearity of Differentiation: $\left(af(x) \pm bg(x)\right)' = af'(x) \pm bg'(x)$

Product Rule (Leibniz Rule): $\left(f(x)g(x)\right)' = f'(x)g(x) + f(x)g'(x)$

Quotient Rule: $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$

Inverse Function Rule: $(f^{-1}(x))' = \frac{1}{f'(x)f^{-1}(x)}$

Reciprocal Rule: $\left(\frac{1}{f(x)}\right)' = \frac{-f'(x)}{\left(f(x)\right)^2}$

Elementary Power Rule: $(x^n)' = nx^{n-1}$, given $n \in \mathbb{Q}$

Generalized Power Rule: $f(x)^{g(x)} = \left(e^{g(x)\ln f(x)}\right)' = f(x)^{g(x)}\left(f'(x)\frac{g(x)}{f(x)} + g'(x)\ln f(x)\right)$

Chain Rule: $\left(f\left(g(x)\right)\right)' = f'\left(g(x)\right)g'(x)$

$\left(e^{f(x)}\right)' = f'(x)e^x$	$(\log_a f(x))' = \frac{f'(x)}{f(x)\log a}$	$\left(a^{f(x)}\right)' = f'(x)a^{f(x)}\ln a$	$(\ln\lvert f(x)\rvert)' = \frac{f'(x)}{f(x)}$
$\sin x' = \cos x$	$\arcsin x' = \frac{1}{\sqrt{1-x^2}}$	$\csc x' = -\csc x \cot x$	$\text{arccsc}\, x' = \frac{1}{\lvert x\rvert\sqrt{1-x^2}}$
$\cos x' = -\sin x$	$\arccos x' = -\frac{1}{\sqrt{1-x^2}}$	$\sec x' = \sec x \tan x$	$\text{arcsec}\, x' = -\frac{1}{\lvert x\rvert\sqrt{1-x^2}}$
$\tan x' = \sec^2 x$	$\arctan x' = \frac{1}{1+x^2}$	$\cot x' = -\csc^2 x$	$\text{arccot}\, x' = -\frac{1}{1+x^2}$

Critical Point (c): a point c : c is in the domain of $f \wedge f'(c) = 0 \vee f'(c) \notin \mathbb{R}$

Stationary Point (c): a point $c : f'(c) = 0$

Given I is an open interval,

Increasing\Decreasing Function: $f'(x)\ \forall x \in I \begin{cases} > 0 \rightarrow increasing\ on\ I \\ = 0 \rightarrow constant\ on\ I \\ < 0 \rightarrow decreaing\ on\ I \end{cases}$

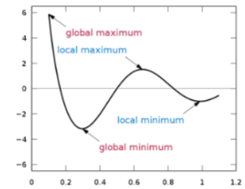
Concavity of a Function: $f''(x)\ \forall x \in I \begin{cases} > 0 \rightarrow concave\ up\ on\ I \\ = 0 \rightarrow possible\ Inflection\ Point \\ < 0 \rightarrow concave\ down\ on\ I \end{cases}$

S.K.C.

First Derivative Test (c is a stationary point): $\begin{cases} f'(x) > 0\ \forall x < c \in I\ \wedge\ f'(x) < 0\ \forall x > c \in I \to local\ minimum\ at\ c \\ f'(x) > 0\ \forall x < c \in I\ \wedge\ f'(x) > 0\ \forall x > c \in I \to neither \\ f'(x) < 0\ \forall x < c \in I\ \wedge\ f'(x) < 0\ \forall x > c \in I \to neither \\ f'(x) < 0\ \forall x < c \in I\ \wedge\ f'(x) > 0\ \forall x > c \in I \to local\ maximum\ at\ c \end{cases}$

Second Derivative Test (c is a stationary point): $f''(c) \begin{cases} > 0 \to local\ minimum\ at\ c \\ = 0 \text{ or } undefined \to possible\ Inflection\ Point \\ < 0 \to local\ maximum\ at\ c \end{cases}$

Extrema: $\begin{cases} Relative\backslash Local \begin{cases} Minima\ (c): f(c) \leq f(x)\ \forall x \in I\backslash\{c\} \\ Maxima\ (c): f(c) \geq f(x)\ \forall x \in I\backslash\{c\} \end{cases} \\ Absolute\backslash Global \begin{cases} Minima\ (c): f(c) \leq f(x)\ \forall x \\ Maxima\ (c): f(c) \geq f(x)\ \forall x \end{cases} \end{cases}$



Fermat's Theorem: $x = c$ is a local extrema of $f(x) \to c$ is a critical point of $f(x)$

Mean Value Theorem: $f(x)$ is continuous on $[a, b] \wedge$ differentiable on $(a, b) \to \exists c \in (a, b): f'(c) = \frac{f(b)-f(a)}{b-a}$

Extended Mean Value Theorem: $f(x) \wedge g(x)$ are continuous on $[a, b] \wedge$ differentiable on $(a, b) \to$ $\exists c \in (a, b): \big(f(b) - f(a)\big)g'(c) = \big(g(b) - g(a)\big)f'(c)$

Rolle's Theorem (Special case of Mean Value Theorem): $f(x)$ is continuous on $[a, b] \wedge$ differentiable on $(a, b) \wedge f(a) = f(b) \to \exists c \in (a, b) :\ f'(c) = 0$

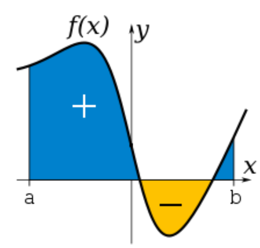
Extreme Value Theorem: $f(x)$ is continuous on $[a, b] \to \exists c \in [a, \infty), d \in (-\infty, b] : f(c)$ is the absolute maximum on $[a, b]$ and $f(d)$ is the absolute minimum on $[a, b]$

Newton's Method: x_n is the n[th] guess for the root/solution of $f(x) = 0 \to$ (n+1)[th] guess $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ provided $f'(x_n)$ exists, and is more accurate

Tangent Line at $x = a$ to $f(x)$: $y = f'(a)(x - a) + f(a)$

Normal\Perpendicular Line at $x = a$ to $f(x)$: $y = -\frac{1}{f'(a)}(x - b) + f(b)$

INTEGRALS:



Definite Integral: Given $f(x)$ is a real-valued continuous function on $[a, b]$ on the real line, dividing $[a, b]$ into n sub-intervals $[x_{i-1}, x_i]$ indexed by i, each of which is "tagged" by the point $t_i \in [x_{i-1}, x_i]$ where $\Delta x_i = x_{i-1} - x_i$ yields the **Reimann integral**, the net signed area of the region in the *xy*-plane bounded by the graph of $f(x)$, the *x*-axis, and the vertical lines $x = a$ and $x = b$.

Definite Integral of $f(x)$ from a to b with respect to x:

$$\int_a^b f(x)\,dx = \lim_{\Delta x_{max} \to 0} \sum_{i=1}^{n} f(t_i)\,\Delta x_i = \lim_{n \to \infty} \sum_{i=1}^{n} f(t_i)\,\Delta x_i$$

Integrability on an Interval: $\lim_{n\to\infty} \sum_{i=1}^{n} f(t_i)\,\Delta x_i$ exists $\to f(x)$ is integrable on $[a, b]$

Continuity on the Interval I $\to$ Integrability on the Interval I

Fundamental Theorem of Calculus: ($f(x) \in \mathbb{R}$ and is continuous $\forall x \in [a, b]$)

1. Let $F(x)$ be the function defined, $\forall x$ on $[a, b]$ by $F(x) = \int_a^x f(t)\,dt$. Then, $F(x)$ is continuous on $[a, b]$, differentiable on (a, b), $\wedge$ $\boldsymbol{F'(x) = f(x)}$ $\forall x \in (a, b)$.
2. Suppose $f(x)$ admits an **antiderivative** $F(x)$ on $[a, b]$. That is, such that $F'(x) = f(x)$. If $f(x)$ is integrable on $[a, b]$, then $\boldsymbol{\int_a^b f(x)\,dx = [f(x)]_a^b = F(b) - F(a)}$.

S.K.C.

Variable of Integration: x is the dummy variable of integration, dx
Antiderivative: $F(x)$ is the antiderivative of $f(x) \leftrightarrow F'(x) = f(x)$
Indefinite Integral: $\int f(x)\,dx = F(x) + C$ where $F(x)$ is the antiderivative of $f(x)$ and C is the **constant of Integration,** an ambiguity that arises within indefinite integrals, $C \in \mathbb{R}$

Given $f(x)$ and $g(x)$ are integrable function and $a, b, c, d, e \in \mathbb{R}$:

Constant Factor Rule: $\int cf(x)\,dx = c\int f(x)\,dx$, $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$
Sum Rule: $\int\big(f(x) + g(x)\big)\,dx = \int f(x)\,dx + \int g(x)\,dx$, $\int_a^b\big(f(x) + g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
Subtraction Rule: $\int\big(f(x) - g(x)\big)\,dx = \int f(x)\,dx - \int g(x)\,dx$, $\int_a^b\big(f(x) - g(x)\big)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$
Linearity of Integration: $\int\big(af(x) \pm bg(x)\big)\,dx = a\int f(x)\,dx \pm b\int g(x)\,dx$, $\int_a^b\big(df(x) \pm eg(x)\big)\,dx = d\int_a^b f(x)\,dx \pm e\int_a^b g(x)\,dx$

Integration by Parts: $\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx$, $\int_a^b f'(x)g(x) = [f(x)g(x)]_a^b - \int_a^b f(x)g'(x)\,dx$
Integration by Substitution (u-substitution): $\int_a^b f\big(g(t)\big)g'(t)\,dt = \int_{g(a)}^{g(b)} f(x)\,dx$

Reversing limits of integration: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
Integrals over intervals of length zero: $\int_a^a f(x)\,dx = 0$
Additivity of integration on intervals: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
Integrals of even functions: $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$
Integrals of odd functions: $\int_{-a}^a f(x)\,dx = 0$

Laite Rule (for Integration by Parts): whichever function comes first should be $g(x)$, last should be $f'(x)$

- L. **Logarithmic functions:** $\ln x$, $\log_b x$
- A. **Inverse trigonometric functions:** $\arctan x$, $\operatorname{arcsec} x$
- I. **Algebraic functions:** $x^2, 3x^{50}$
- T. **Trigonometric functions:** $\sin x$, $\tan x$
- E. **Exponential functions:** $e^x, 19^x$

$$\int e^x\,dx = e^x + C \qquad \int a^x\,dx = \frac{a^x}{\ln a} + C \qquad \int \ln x\,dx = x\ln x - x + C \qquad \int \log_a x\,dx = x\log_a x - \frac{x}{\ln a} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C \qquad \int \tan x\,dx = \ln|\sec x| + C$$

$$\int \csc x\,dx = -\ln|\cos x + \cot x| + C \qquad \int \sec x\,dx = \ln|\sec x + \tan x| + C \qquad \int \cot x\,dx = \ln|\sin x| + C$$

Composite Newton-Cotes Formulas: quadrature techniques based on interpolating functions ($a = x_o \leq \xi \leq b = x_n$), formulas below are given N equally spaced partitions

1. **Midpoint Rule (0 point – open):** $\int_a^b f(x)dx = \sum_{i=1}^N \frac{b-a}{N} f\left(\frac{x_i + x_{i-1}}{2}\right) + \frac{(b-a)^3}{24N^2} f''(\xi)$
2. **Trapezoidal Rule (1 point – closed):** $\int_a^b f(x)dx = \sum_{i=1}^N \frac{b-a}{2N}\big(f(x_{i-1}) + f(x_i)\big) - \frac{(b-a)^3}{12N^2} f''(\xi) =$
   $\frac{b-a}{2N}\big(f(x_0) + 2f(x_1) + 2f(x_2) \dots 2f(x_{n-1}) + f(x_n)\big) - \frac{(b-a)^3}{12N^3} f''(\xi)$
3. **Simpson's Rule (2 point – closed):** $\int_a^b f(x)dx = \sum_{i=2}^N \frac{b-a}{3N}\left(f(x_{i-2}) + 4f\left(\frac{x_i + x_{i-2}}{2}\right) + f(x_i)\right) - \frac{(b-a)^5}{180N^2} f^{(4)}(\xi) =$
   $\frac{b-a}{3N}\big(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) \dots 4f(x_{n-1}) + f(x_n)\big) - \frac{(b-a)^5}{180N^2} f^{(4)}(\xi)$

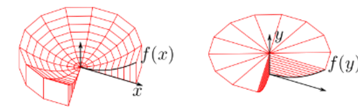
S.K.C.

Mean Value of a Function over Interval (a, b): $\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$

Net Signed Area of a Function over Interval (a, b): $A = \int_a^b f(x)\,dx$

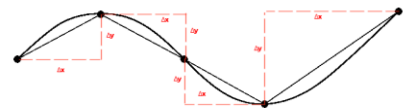
Solids of Revolution: (where h\v is the horizontal\vertical axis of rotation)

1. **Shell\Cylinder Method of Integration:** $V = 2\pi\int_a^b x|f(x) - g(x)|\,dx$
2. **Disk\Washer Method of Integration:** $V = \pi\int_a^b |f(x)^2 - g(x)^2|\,dx$



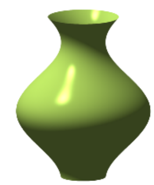
Arc Length:

1. **Rectangular Equation ($y = f(x)$):** $s = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$
2. **Parametric Equation ($x = f(t), y = g(t)$):** $s = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt$
3. **Polar Equation ($r = f(\theta)$):** $s = \int_a^b \sqrt{r^2 + [f'(\theta)]^2}\,d\theta$



Surface of Revolution:

1. **Parametric Equation over x-axis ($x = f(t), y = g(t)$):** $A_y = 2\pi\int_a^b f(t)\sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt$
2. **Parametric Equation over y-axis ($x = f(t), y = g(t)$):** $A_y = 2\pi\int_a^b g(t)\sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt$
3. **Rectangular Equation ($y = f(x)$):** $A_y = 2\pi\int_a^b f(x)\sqrt{1 + [f'(x)]^2}\,dx$



SEQUENCE & SERIES

Sequence: $\{a_n\}: a_0, a_1, a_2 \ldots a_n \in \mathbb{R}, n \in \mathbb{N}, f(x) = a_n \forall x \in \mathbb{N}$

Limit of a Sequence: $\lim_{n\to\infty} a_n = L \in \mathbb{R} \leftrightarrow \forall\varepsilon > 0\ \exists\ N \in \mathbb{N} : \forall n\ (n \geq N \to |a_n - L| < \epsilon)$

Convergence\Divergence of a Sequence: $\begin{cases} \lim_{n\to\infty} a_n = \lim_{n\to\infty} f(x) = L \in \mathbb{R} \leftrightarrow \{a_n\}\ converges \\ \lim_{n\to\infty} |a_n| = \lim_{x\to\infty} |f(x)| = L \in \mathbb{R} \leftrightarrow \{a_n\}\ converges\ absolutely \\ \lim_{n\to\infty} a_n \neq L \in \mathbb{R} \leftrightarrow \{a_n\}\ diverges \end{cases}$

Series: $\{S_N\}: S_n = \sum_{i=0}^{n} a_i, n \in \mathbb{N}$

Sum of a Series: $\sum_{n=0}^{\infty} a_n = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{n=0}^{N} a_n \begin{cases} \lim_{N\to\infty} S_N = L \in \mathbb{R} \leftrightarrow \{S_N\}\ converges \\ \lim_{N\to\infty} S_N \neq L \in \mathbb{R} \leftrightarrow \{S_N\}\ diverges \end{cases}$

Monotonic Sequence: $a_i \leq a_j\ \forall j > i \vee a_i \geq a_j\ \forall j > i$

Bounded Sequence: $\begin{cases} \exists M : a_n \leq M \leftrightarrow \{a_n\}\ bounded\ above\ by\ M \\ \exists N : a_n \leq N \leftrightarrow \{a_n\}\ bounded\ below\ by\ N \\ \exists M : a_n \leq M \wedge \exists N : a_n \leq N \leftrightarrow \{a_n\}\ bounded \end{cases}$

Monotonic and Bounded Sequence: $\left(a_i \leq a_j\ \forall j > i \vee a_i \geq a_j\ \forall j > i\right) \wedge \exists M : a_n \leq M \wedge \exists N : a_n \leq N \to \lim_{n\to\infty} a_n = L \in \mathbb{R}$

Geometric Series (ratio r): $\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots + ar^n + \cdots = \frac{a}{1-r}\ (converges) \leftrightarrow |r| < 1$

nth Term Test for Divergence: $\lim_{n\to\infty} a_n \neq 0 \vee \lim_{n\to\infty} a_n \neq L \in \mathbb{R} \to \{S_N\}\ diverges$

Integral Test for Convergence: $\{S_N\}\ converges \leftrightarrow \int_0^{\infty} f(x)\,dx = L \in \mathbb{R}, f(x) \geq 0 \wedge f(x) = a_n \forall x \in \mathbb{N} \wedge f'(x) < 0\ \forall x > 0$

p-Series: $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots\ converges \leftrightarrow p > 1 \neg p > 1 \leftrightarrow diverges$

Harmonic Series: $\sum_{n=0}^{\infty} \frac{1}{an+b}\ diverges, a \neq 0 \wedge b \in \mathbb{R}$

Telescoping Series: a series whose partial sums eventually only have a fixed number of terms after cancellation

Direct Comparison Test: $0 < a_n \leq b_n \forall n > N \in \mathbb{N} \begin{cases} \sum_{n=0}^{\infty} b_n\ converges \to \sum_{n=0}^{\infty} a_n\ converges \\ \sum_{n=0}^{\infty} a_n\ diverges \to \sum_{n=0}^{\infty} b_n\ diverges \end{cases}$

Limit Comparison Test: $a_n > 0 \wedge b_n > 0\ \forall n > N \in \mathbb{N}, \lim_{n-\infty} \frac{a_n}{b_n} = L \in \mathbb{R} > 0 \to \sum_{n=0}^{\infty} a_n \wedge \sum_{n=0}^{\infty} b_n\ both\ converge \vee diverge$

Alternating Series Test: $a_n > 0, \sum_{n=0}^{\infty}(-1)^n a_n \wedge \sum_{n=0}^{\infty}(-1)^{n+1} a_n \; converge \leftrightarrow \lim_{n\to\infty} a_n = 0 \wedge a_{n+1} \le a_n \forall n > N \in \mathbb{R}$

Ratio Test: $a_n \neq 0 \; \forall n \in \mathbb{N}, \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \begin{cases} < 1 \to \sum_{n=0}^{\infty} a_n \; converges\ absolutely \\ = \infty \vee > \; 1 \to \sum_{n=0}^{\infty} a_n \; diverges \\ = 1 \to inconclusive \end{cases}$

Root Test: $\lim_{n\to\infty} \sqrt[n]{|a_n|} \begin{cases} < 1 \to \sum_{n=0}^{\infty} a_n \; converges\ absolutely \\ = \infty \vee > \; 1 \to \sum_{n=0}^{\infty} a_n \; diverges \\ = 1 \to inconclusive \end{cases}$

nth Taylor Polynomial of degree n for $f(x)$ given $f(x)$ is differentiable in a neighborhood of a real or complex number a:

$$f(x) = P_{n,a} = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

$: R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} : \xi \in (x, a)$ is the Lagrange form of the remainder of degree n

In the case that $a = 0$, the polynomial is also called a nth Maclaurin polynomial of degree n and denoted P_n

Taylor Series for $f(x)$ given $f(x)$ is infinitely differentiable in a neighborhood of a real or complex number a:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots$$

In the case that $a = 0$, the series is also called a Maclaurin series

Power Series centered at a with a radius of convergence r : $\forall x \in (a-r, a+r)\; f(x)\; converges$:

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n = a_0 + a_1(x-a)^1 + a_2(x-a)^2 + a_3(x-3)^2 + \cdots$$

Binomial Series centered at 0 with a radius of convergence $1 \wedge \alpha \in \mathbb{R}$

$$(1+x)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n = 1 + ax + \frac{a(a-1)x^2}{2!} + \frac{a(a-1)(a-2)x^3}{3!} + \frac{a(a-1)(a-2)(a-3)x^4}{4!} \ldots$$

S.K.C.